AARON'S MATH GUIDE FOR THE ISEE UPPER LEVEL

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1 Reducing Fractions by Common Factors

- If a fraction has a common factor in both the numerator and denominator you can divide both of them by the factor.
- Take the fraction $\frac{10}{24}$. Both of these are even numbers, so we know they are divisible by two. We then divide 10 by 2 and 24 by 2. This gives us $\frac{5}{12}$, the fully reduced form.
- Dividing the numerator and the denominator by the same number is equal to multiplying by 1. That is why reducing a fraction doesn't change its value.
- The order in which you choose to reduce a fraction doesn't matter. See "Order of Operations."
- Another example: $\frac{10 \cdot 15 \cdot 6}{5 \cdot 3 \cdot 4}$
  - ◇ Choose a point of entry and start. We will start by dividing the 10 in the numerator and 5 in the denominator each by 5.
    $= \frac{2 \cdot 15 \cdot 6}{1 \cdot 3 \cdot 4}$ *now divide* 15 *and* 3 *by* 3
    $= \frac{2 \cdot 5 \cdot 6}{1 \cdot 1 \cdot 4}$ *now divide* 6 *and* 4 *by* 2
    $= \frac{2 \cdot 5 \cdot 3}{1 \cdot 1 \cdot 2}$ *now divide* 2 *and* 2 *by* 2
    $= \frac{1 \cdot 5 \cdot 3}{1 \cdot 1 \cdot 1}$ *simplify*
    $= 15$
- We can also expand fractions with a common denominator:

  $$\frac{x^2 + 3x + 10}{x} = \frac{x^2}{x} + \frac{3x}{x} + \frac{10}{x} = x + 3 + \frac{10}{x}$$

  $$\frac{z^2 + 4zx + 12\sqrt{z}}{z^2} = \frac{z^2}{z^2} + \frac{4xz}{z^2} + \frac{\sqrt{z}}{z^2} = 1 + \frac{4x}{z} + z^{-\frac{3}{2}}$$

  $$\frac{(x+3)(x-4) + (x+8)(x^2+10)(x-4)}{(x-4)} = \frac{(x+3)(x-4)}{(x-4)} + \frac{(x+8)(x^2+10)(x-4)}{(x-4)} = (x+3) + (x+8)(x^2+10)$$

  ⚠ We <u>cannot</u> expand in the same way using a denominator.

  $$\frac{x}{x-4} \neq \frac{x}{x} - \frac{4}{x}$$

  ⚠ If there is addition or subtraction in the fraction you cannot cross out a number unless it is a factor of every single other number in the fraction.
  - ◇ $\frac{5+23}{5} \neq 1 + 23$
  - ◇ $\frac{8+15}{4+5} \neq 2 + 3$
  - ◇ $\frac{12+22+6}{2+12} = \frac{6+11+3}{1+6}$ *must divide every group of addition or subtraction by* 2
  - ◇ $\frac{(4)(9)+(3)(8)+(6)(7)}{3} = \frac{(4)(3)+(1)(8)+(2)(7)}{1}$ *divide chunks* (4)(9), (3)(8), *and* (6)(7) *each by* 3

2 Percentage Point "Chunking"

- Break up a tough percentage to a combo of several smaller ones:

$50\% \text{ of } x = \frac{x}{2}$	$25\% \text{ of } x = \frac{50\% \text{ of } x}{2}$	$10\% \text{ of } x = \frac{x}{10}$	$5\% \text{ of } x = \frac{10\% \text{ of } x}{2}$	$1\% \text{ of } x = \frac{10\% \text{ of } x}{10}$

- 19%
  $= 20\% - 1\%$
  $= 10\% + 10\% - 1\%$
- 71%
  $= 70\% + 1\%$
  $= 50\% + 10\% + 10\% + 1\%$
- 136%
  $= 100\% + 30\% + 6\%$
  $= 100\% + 10\% + 10\% + 10\% + 5\% + 1\%$
- Percentages in the tens can sometimes be calculated easier using fractions:

$$30\% \text{ of } x = \frac{x \cdot 3}{10}$$
$$60\% \text{ of } x = \frac{x \cdot 6}{10}$$
$$20\% \text{ of } x = \frac{x \cdot 2}{10}$$

- The main idea can be applied to other problems; if you're having trouble, try separating things into smaller pieces that are easier to deal with!

3 Changes in Percent

- Whenever you need to find the *Percent Change* in something, you will want to do the following:

$$\text{Percent Change} = \frac{\textit{Final Value - Initial Value}}{\textit{Initial Value}}$$

  - ◇ If the percent change is positive, we can say there was a *Percent Increase.*
  - ◇ If the percent change is negative, we can say there was a *Percent Decrease.*
- Sales figures went up from 300 to 400 last month. Calculate the percent increase in sales.
  - ◇ Final = 400, Initial = 300

$$\text{Percent Change} = \frac{400 - 300}{300} = \frac{100}{300} = \frac{1}{3} \approx +33\%$$

- The number of attendees at a play this year decreased by 20% from last year. If 250 people went to the play last year, how many went this year?
  - ◇ Final = ?, Initial = 250, Percent Change = −20%

$$-20\% = \frac{F - 250}{250}$$
$$\Rightarrow -0.2 = \frac{F - 250}{250}$$
$$\Rightarrow -\frac{2}{10} = \frac{F - 250}{250}$$
$$\Rightarrow -\frac{2 \cdot 250}{10} = F - 250$$
$$\Rightarrow -(2 \cdot 25) = F - 250$$
$$\Rightarrow F = -50 + 250 = 200$$

- 25% more people preferred foodcourt X over foodcourt Y. If 1200 people preferred Y, how many preferred X?

- ◇ Final = ?, Initial = 1200, Percent Change = +25%

  $25\% = \dfrac{P - 1200}{1200}$

  $\Rightarrow 25\% \cdot 1200 = P - 1200$

  $\Rightarrow \dfrac{1200}{4} = P - 1200$

  $\Rightarrow 300 = P - 1200$

  $\Rightarrow P = 1500$

4 Equations from Words

- The ISEE has a lot of word problems. You want to be good at forming mathematical relationships from the given questions.
- There are some common tells that let you know there is a mathematical relationship going on.
- Look for words that compare things such as "more", "less", "greater", "twice", "half", "sum", "difference", "product" (multiply), and "quotient" (divide).
- Example: b is 5 more than a. c is three more than the sum of a and b. If $a = 2$, what is c?
  - ◇ Here our words are "more" and "sum".
  - ◇ We use them to form our equations:

    $b = a + 5$ *b is five more than a*

    $c = 3 + (a + b)$ *c is three more than the sum of a and b*
  - ◇ We can then substitute and find the answer:

    $b = 2 + 5 = 7$

    $c = 3 + (2 + 7) = 12$
- A more complicated example: There are 4 chocolate chip cookies in a cookie jar. Twice the number of chocolate chip cookies is equal to four times the number of gingersnaps in the jar. 3 more than the difference of the chocolate chip cookies and gingersnaps is equal to one-half the number of snickerdoodles in the jar. How many cookies are in the jar?
- Here is a good time to apply the "Chunking" technique. We will go one sentence at a time.
- Let's also get symbols for the cookies we are working with, because writing them out every time takes a while.

  $C = chocolate\ chip$

  $G = gingersnap$

  $S = snickerdoodle$
- To begin:

  $C = 4$

  $2 \cdot C = 4 \cdot G$ *twice C, four times G*

  $3 + (C - G) = \dfrac{S}{2}$ *three more, difference of C and G, one-half S*
- Now we can substitute and solve:

  $2 \cdot 4 = 4 \cdot G \Rightarrow G = 2$

  $3 + (4 - 2) = \dfrac{S}{2} \Rightarrow 5 = \dfrac{S}{2} \Rightarrow S = 10$
- So the total number of cookies $= C + G + S = 4 + 2 + 10 = 16$

5 FOIL

- How would you expand $(x+3)(x-4)$, equivalent to $(x+3)\cdot(x-4)$
- FOIL stands for First, Outer, Inner, Last
- This refers to the terms that you multiply. You then take the sum of all of your terms to get the answer. For the above example:

  $(x+3)(x-4)$
  $FIRST$: $x \cdot x = x^2$
  $OUTER$: $x \cdot (-4) = -4x$
  $INNER$: $3 \cdot x = 3x$
  $LAST$: $3 \cdot -4 = -12$

  ◇ To get our final answer we then take the sum of all four terms:
  $= x^2 + (-4x) + 3x + (-12)$
  $= x^2 - x - 12$

- A more difficult example: $(5-z^2)(\sqrt{x}+2)$

  FIRST: $5 \cdot \sqrt{x} = 5\sqrt{x}$
  OUTER: $5 \cdot 2 = 10$
  INNER: $-z^2 \cdot \sqrt{x} = -z^2\sqrt{x}$
  LAST: $-z^2 \cdot 2 = -2z^2$
  $= 5\sqrt{x} + 10 + (-z^2\sqrt{x}) + (-2z^2)$
  $= 5\sqrt{x} + 10 - z^2\sqrt{x} - 2z^2$

- This concept can be expanded to situations were we have different numbers of terms.

  ◇ This general idea is to take the term from the first group and multiply it by all the terms from the second group, summing each result. Then you move to the next term in the first group until you hit the last term.
  $(3a+b)(a^2+2ab+4)$

  ◇ First term of first group $3a$:
  $= 3a \cdot a^2 + 3a \cdot 2ab + 3a \cdot 4$
  $= 3a^3 + 6a^2b + 12a$

  ◇ Second term of first group b:
  $= b \cdot a^2 + b \cdot 2ab + b \cdot 4$
  $= a^2b + 2b^2a + 4b$

  ◇ Sum both together:
  $= 3a^3 + 6a^2b + a^2b + 2b^2a + 12a + 4b$
  $= 3a^3 + 7a^2b + 2b^2a + 12a + 4b$

- Finally, what about this:

  $(x+3)(x+4)(x-5)$

  ◇ The best thing to do here is to chunk it into pieces. We will go left to right, starting with $(x+3)(x+4)$:
  $= x^2 + 4x + 3x + 12$
  $= x^2 + 7x + 12$

  ◇ Now we expand the above answer with $(x-5)$:
  $= (x^2 + 7x + 12)(x-5)$
  $= x^3 - 5x^2 + 7x^2 - 35x + 12x - 60$
  $= x^3 + 2x^2 - 23x - 60$

6 Unit Conversions, AKA "Railroad Tracks"

- The key to unit conversion problems is to know the units of where you start and the units of where you end. Then use the given info to convert your starting point.
- If you don't see an immediate bridge, try looking for associations between the two. For example, if you need to find a rate of gallons/min and you're starting with gallons/hour, you can convert your hours to minutes.
- Example: Convert 240 minutes to time in days.

$$\frac{240 \text{ minutes}}{} \Big| \frac{1 \text{ hour}}{60 \text{ minutes}} \Big| \frac{1 \text{ day}}{24 \text{ hours}}$$
$$= \frac{240 \text{ \sout{minutes}}}{} \Big| \frac{1 \text{ \sout{hour}}}{60 \text{ \sout{minutes}}} \Big| \frac{1 \text{ day}}{24 \text{ \sout{hours}}}$$
$$= \frac{240}{(60)(24)} \text{ days}$$
$$= \frac{1}{6} \text{ days}$$

- A bit harder: One flim-flam is equal in value to 10 dados. 5 dados are equal to 12 pling-plangs. Find the value of 3 flim-flams in terms of pling-plangs.

$$\frac{3 \text{ flim-flams}}{} \Big| \frac{10 \text{ dados}}{\text{flim-flam}} \Big| \frac{12 \text{ pling-plangs}}{5 \text{ dados}}$$
$$\frac{3 \text{ \sout{flim-flams}}}{} \Big| \frac{10 \text{ \sout{dados}}}{\text{\sout{flim-flam}}} \Big| \frac{12 \text{ pling-plangs}}{5 \text{ \sout{dados}}}$$
$$= \frac{(3)(10)(12)}{5} \text{ pling-plangs}$$
$$= 3 \cdot 2 \cdot 12 \text{ pling-plangs}$$
$$= 72 \text{ pling-plangs}$$

7 Formulae

- Sum of the interior angles of a polygon $= 180(n-2)$, where n is the number of sides.
  - If you're having trouble remembering then think of a triangle. The angles add up to 180° and there are three sides.
- Area of a trapezoid $A = \frac{1}{2} \cdot (b_1 + b_2) \cdot h$
  - Think of a trapezoid as a triangle with two bases. Instead of $\frac{1}{2}bh$, you now have a second base you need to add before you multiply by the height.

8 Exponent Operations

- Mathematicians are lazy, so exponents are just a shorthand for multiplication.
- $4^3 = 4 \cdot 4 \cdot 4$
- $3^5 = (3)(3)(3)(3)(3)$, and so on
- The ISEE will have you do math with exponents. You need to remember their properties listed below.
- Division of Like Terms:
  - When you are dividing like terms with exponents, the resulting exponent will always be the difference of the numerator and denominator (subtract them).

$$\frac{x^5}{x^2} = x^{5-2} = x^3$$
$$\frac{x^{-3}}{x^6} = x^{-3-6} = x^{-9}$$

$\frac{z^4}{z^{-12}} = z^{4-(-12)} = z^{4+12} = z^{16}$

$\frac{y^{1/2}}{y^{2/3}} = y^{\frac{1}{2}-\frac{2}{3}} = y^{\frac{3-4}{6}} = y^{-\frac{1}{6}} = \frac{1}{\sqrt[6]{y}}$

$\frac{q^3}{r^5} =$ *These are two different terms, so we cannot simplify any further.*

- Multiplication of Like Terms:
  - ◇ When you are multiplying two exponents with like terms, you add them together.

    $x^3 \cdot x^2 = x^{3+2} = x^5$

    $p^{-3} \cdot p^{18} = p^{-3+18} = p^{21}$

    $(\varphi^2)(\varphi^{-5}) = \varphi^{2+(-5)} = \varphi^{-3} = \frac{1}{\varphi^3}$

    $x^{\frac{1}{6}} \cdot x^{\frac{1}{3}} \cdot x^4 = x^{\frac{1}{6}+\frac{1}{3}+4} = x^{\frac{1}{6}+\frac{2}{6}+\frac{24}{6}} = x^{\frac{27}{6}} = x^{\frac{9}{2}}$

- Exponentiation of an Exponent:
  - ◇ When you calculate an exponent of a term that already has an exponent, you multiply the two exponents together.

    $x^{2^3} = x^{2 \cdot 3} = x^6$

    $z^{(\frac{1}{2})^3} = z^{\frac{3}{2}}$

    $\Omega^{-3^{-5}} = \Omega^{(-3)(-5)} = \Omega^{15}$

- Taking a Reciprocal (Inverse):
  - ◇ When you invert an exponent, you swap its sign.

    $\frac{1}{x^3} = x^{-3}$

    $\frac{1}{x^{-\frac{27}{112}}} = x^{\frac{27}{112}}$

  - ◇ Consequently, answer choices will often be options with all positive exponents. Know how they got reduced.

    $x^8 z^{-12} q^{15} \theta^{-8} = \frac{x^8 q^{15}}{z^{12} \theta^8}$

- Roots:
  - ◇ $x^{\frac{1}{2}}$ is defined as the "square root" of x; $\sqrt{x}$
  - ◇ $x^{\frac{1}{3}}$ is defined as the "cube root" of x; $\sqrt[3]{x}$
  - ◇ $x^{\frac{1}{4}}$ is the "fourth root"; $\sqrt[4]{x}$
  - ◇ $x^{\frac{1}{5}}$ is the "fifth root"; $\sqrt[5]{x}$
  - ◇ and so on...
  - ◇ Don't be scared by the fractions! Their properties are the same as whole numbers.

- Addition and Subtraction of Like Terms:
  - ◇ There is no quick trick to add and subtract exponential like terms.
  - ◇ Look at 5 and 5^2. We know 5^3 is 125. Do you think 5 and 5^2 add up to 125? Or that $25 - 5 = 5$?

    $x^3 + x^8 \neq x^{13}$

    $q^8 - q^2 \neq q^6$

- Powers of Zero and One:
  - Any number without an exponent has an assumed “to the power of one.”
    $2 = 2^1$
    $8 = 8^1$
    $z = z^1$
    $\frac{x^4}{x} = x^{4-1} = x^3$
  - Any number (except 0) to the power of 0 equals 1. 0^0 is undefined.
    $5^0 = 0$
    $1225512098^0 = 0$
    $(xyzqe^5\varphi\Lambda\gamma^{12})^0 = 0$ *(provided none of the variables are zero, then it would be undefined)*
  - 1 to the power of any number is 1.
    $1^{235123} = 1$
    $1^{-100} = 1$

9 Order of Operations

- The Order of Operations is commonly known as PEMDAS, with the letters meaning which operators you calculate first.
- A good way to remember PEMDAS:
  - Please Excuse My Dear Aunt Sally.
  - Please Exhume My Dead Aunt Sally.
  - Whichever you prefer!
- Parentheses, Exponentiation, Multiplication and Division, Addition and Subtraction
- Notice how Multiplication and Division are grouped together. As are Addition and Subtraction.
- This is because as long as you don’t do any other operations you can calculate these in any order.
- Example: (5)(3)(2)(4)
  - One Way:
    $= 5 \cdot 3 \cdot 2 \cdot 4$
    $= 15 \cdot 2 \cdot 4$
    $= 30 \cdot 4$
    $= 120$
  - Another Way:
    $= 5 \cdot 3 \cdot 2 \cdot 4$
    $= 5 \cdot 3 \cdot 8$
    $= 5 \cdot 24$
    $= 120$
    The same answer.
- Also true for addition and subtraction, as stated before. Try it yourself!
- But lets violate this rule and see what happens:

  $8 \cdot 2 - 4$
  - Prioritizing the subtraction
    $= 8 \cdot -2$
    $= -16$
    Wrong.

- ⋄ The correct answer is following PEMDAS and doing the multiplication first:
  $= 16 - 4$
  $= 12$
  Correct.

- What about $8 \cdot 2 - 4^2$?
  - ⋄ Remember, exponents come <u>before</u> subtraction. Do not square negative 4; square positive 4.
    $= 8 \cdot 2 - 16$
    $= 16 - 16$
    $= 0$
    Correct.
    $= 8 \cdot 2 + 16$
    $= 16 + 16$
    $= 32$
    Incorrect.

10 $-x^2 = -(x^2)$

- The real number $-x^2$ will always be negative.
- This holds true for any even power of x.
- x^2 is always computed first, and then the negative is applied after.
- This is because of the order of operations (PEMDAS).

  let $x = 5$: $-x^2 = -(5^2) = -(5)(5) = -25$
  let $x = -5$: $-x^2 = -((-5)^2) = -(-5)(-5) = -25$

11 Operators

- Operators are the symbols for things such as addition $(x + y)$, subtraction $(x - y)$, multiplication $(x \cdot y$, also $x \times y)$, division $(\frac{x}{y}$, also $x \div y)$, and exponentiation (x^y). The symbols are representations of the math you perform.
- Sometimes the ISEE will define a new operator for you, and then expect you to do a sample calculation.
- This will always be in terms of operators you already know.
- For instance: $a \gamma b = a^2 + 2b - 6$
  - ⋄ Here, the symbol γ is the operator.
  - ⋄ This symbol is the Greek letter Gamma, but if you don't know how to say a symbol just make up a word for it!
  - ⋄ This means that the γ for any two numbers a and b is equal to $a^2 + 2b - 6$
  - ⋄ So, for $5 \gamma 2$
    $= 5^2 + 2 \cdot 2 - 6$
    $= 25 + 4 - 6$
    $= 25 - 2$
    $= 23$
- Let's have another: $x \xi y = 5x - 2y$
  - ⋄ Here ξ is the operator.
  - ⋄ So for $2 \xi 3$
    $= 5 \cdot 2 - 2 \cdot 3$
    $= 10 - 6$
    $= 4$

- A more difficult example:
  - ◇ $(3 \gamma 2) \xi 10$
  - ◇ Parentheses will always take priority, so we calculate $3 \gamma 2$ first.
    $= 3^2 + 2 \cdot 2 - 6$
    $= 9 + 4 - 6$
    $= 7$
  - ◇ Now, we substitute 7 in the place of $3 \gamma 2$; this will lead us to calculate $7 \xi 10$
    $= 5 \cdot 7 - 2 \cdot 10$
    $= 35 - 20$
    $= 15$
- Operators can also be defined in terms of only one number.

  $\boxed{q} = q^2 + \sqrt{q} + 6$

  $\boxed{4} = 4^2 + \sqrt{4} + 6 = 16 + 2 + 6 = 24$

12 Function Notation

- Functions are similar to operators.
- Take the function $f(x) = x^2 + 9$
  - ◇ $f(x)$ is describing what an output will be when given a certain input x.
  - ◇ So $f(5) = 5^2 + 9 = 25 + 9 = 34$
- Now saw we have some function $g(z) = \sqrt{z} - 2z + 14$
  - ◇ You will sometimes find questions asking for $g(f(x))$ or $f(g(x))$
  - ◇ Let's find $g(f(4))$
  - ◇ To do this, we first find $f(4)$ and then plug it into our function g:
    $f(4) = 4^2 + 9 = 16 + 9 = 25$
  - ◇ Now we plug the number 25 as our z into $g(z)$:
    $= \sqrt{25} - 2(25) + 14$
    $= 5 - 50 + 14$
    $= -31$

13 Matrices

- Matrices are a way to structure data. They have many uses in mathematics, but for the ISEE you will only need to know how to add/subtract them and multiply them by a constant.
- This is a basic matrix structure:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Matrices are broken down into *rows* and *columns.*
- Rows go left to right.
- Columns go up and down.
- The example matrix has three columns and three rows.
- The notation a_{11} means some number a that is in the first row and first column.

- a_{32} means the number a that is in the third row and second column.
- a_{23} means the number a that is in the second row and third column.
- To add matrices, you add the numbers that are in the same row and column. Note that this means you cannot add matrices that are different sizes!

$$\begin{pmatrix} 1 & x & y \\ 2 & 6 & -8 \\ 14 & -5 & 10 \end{pmatrix} + \begin{pmatrix} y & x^2 & 1 \\ -2 & 11 & 6 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1+y & x+x^2 & y+1 \\ 2+(-2) & 6+11 & -8+6 \\ 14+3 & -5+4 & 10+5 \end{pmatrix} = \begin{pmatrix} 1+y & x+x^2 & y+1 \\ 0 & 17 & -2 \\ 17 & -1 & 15 \end{pmatrix}$$

- Subtracting matrices is a similar process.

$$\begin{pmatrix} x & 3 & 0 \\ 4 & z & -12 \\ z^2 & 2 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 5 & 2 \\ -2 & z & 3 \\ 4z^2 & 1 & x \end{pmatrix} = \begin{pmatrix} x-(-1) & 3-5 & 0-2 \\ 4-(-2) & z-z & -12-3 \\ z^2-4z^2 & 2-1 & 6-x \end{pmatrix} = \begin{pmatrix} x+1 & -2 & -2 \\ 6 & 0 & -15 \\ -3z^2 & 1 & 6-x \end{pmatrix}$$

- Multiplying a matrix by a constant is easy. You just distribute it to every number in the matrix.

$$5\begin{pmatrix} \omega & x & 0 \\ x & \beta & -4 \\ y^2 & 4 & \sqrt{z} \end{pmatrix} = \begin{pmatrix} 5\omega & 5x & 0 \\ 5x & 5\beta & -20 \\ 5y^2 & 20 & 5\sqrt{z} \end{pmatrix}$$